Decomposition of geometric perturbations

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(Dated: November 12, 2008)

Abstract

For an infinitesimal deformation of a Riemannian manifold, we prove that the scalar, vector, and tensor modes in decompositions of perturbations of the metric tensor, the scalar curvature, the Ricci tensor, and the Einstein tensor decouple if and only if the manifold is Einstein. Four-dimensional space-time satisfying the condition of the theorem is homogeneous and isotropic. Cosmological applications are discussed.

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I. INTRODUCTION

The theory of small perturbations in general relativity [1] is an important tool in contemporary cosmology. Since the observed universe is almost homogeneous and isotropic, the unperturbed space-time is usually assumed to have the Robertson-Walker form. Although this is a good approximation, recent cosmological observations suggest the necessity of exploring beyond it. Without the assumptions of homogeneity and isotropy, a variety of complications arise, one of which is the mixing of various perturbation modes. Here we explore this issue.

Let M be a manifold with the metric tensor g. For an infinitesimal deformation of M with the perturbation of the metric tensor δg, the perturbation of the Ricci tensor δR is a linear functional of δg. It is convenient to decompose δg into irreducible parts relative to g. This leads to the decomposition of δR. In general, the decompositions of δg and δR are coupled. (The precise definition is given in the next section.) This is due to the fact that local preferred directions specified by g are only a subset of directions specified by the unperturbed Ricci tensor R and the unperturbed Weyl tensor W. The purpose of this note is to find necessary and sufficient conditions on g for which the decompositions of δg and δR decouple, and its implications for cosmology.

A manifold of constant curvature is an example for which the decompositions decouple. Manifolds of constant curvature are classified and are used as model spaces for various constructions in differential geometry. For such a manifold, $R = \frac{1}{n}Sg$ and $W = 0$, where $n = \dim M$ and the constant S is the scalar curvature of M. As a result, the only preferred directions in this case are those specified by g. This example suggests that the key property needed for the decompositions of δg and δR to decouple is an appropriate generalization of a manifold of constant curvature.

A natural generalization of a manifold of constant curvature is an Einstein manifold. For such a manifold, $R = \frac{1}{n}Sg$, where S is a constant. Properties of W imply that it can enter a relation between the terms of the tensor type only, which prevents W from coupling the decompositions. In two and three dimensions, the only tensor with the required properties of the Weyl tensor is the zero tensor. This implies that a manifold of dimension two or three is Einstein if and only if it is a manifold of constant curvature.

As our main result, we prove a theorem which states, in particular, that the decomposi-

tions of δg and δR decouple if and only if M is an Einstein manifold.

The above considerations have application to geometric theories of gravity. In such a theory, the action functional is $I + J$, where I and J are the geometric and matter parts of the action, respectively. The equation of motion is $G + T = 0$, where G and T are the variational derivatives of I and J, respectively, with respect to g. G is a generalization of the Einstein tensor and T is the energy-momentum tensor. In this case, we are interested in the decompositions of δg and δG. When the decompositions decouple, the equation of motion reduces to separate equations for each irreducible part, which usually simplifies their solutions. When the decompositions are coupled, it is usually harder to solve the equations, but in such cases there are interesting classes of solutions where perturbations of one type in δT leads to perturbations of multiple types in δg.

II.   DECOMPOSITION THEOREM

Let M be an n-dimensional manifold with metric g and connection ∇.

Definition 1. *Let h be a symmetric tensor of type $(0,2)$ on (M, g, ∇). The set $(\varphi, \psi, \theta, \omega)$ is called the decomposition of the tensor h if*

$$h_{ij} = \tfrac{1}{n}\varphi g_{ij} + 2(\nabla_i \nabla_j - \tfrac{1}{n}g_{ij}\Delta)\psi + \nabla_i\theta_j + \nabla_j\theta_i + \omega_{ij},\tag{1}$$

where

$$\nabla^i\theta_i = 0,\quad \omega^i{}_i = 0,\quad \nabla^j\omega_{ij} = 0.\tag{2}$$

We obtain expression (1) from the decomposition [2]

$$h_{ij} = \tfrac{1}{n}\varphi g_{ij} + \nabla_i\xi_j + \nabla_j\xi_i - \tfrac{2}{n}g_{ij}\nabla_k\xi^k + \omega_{ij},\tag{3}$$

where φ is a scalar, ξ is a vector, and ω is a symmetric, traceless, and transverse tensor of type $(0,2)$. It follows that $\varphi = h^i{}_i$, and the condition $\nabla^j\omega_{ij} = 0$ then gives

$$\nabla^j\nabla_i\xi_j + \Delta\xi_i - \tfrac{2}{n}\nabla_i\nabla^j\xi_j = \nabla^j(h_{ij} - \tfrac{1}{n}\varphi g_{ij}).\tag{4}$$

The solution ξ of Eq. (4) is unique up to a conformal Killing vector η, which satisfies the equation

$$\nabla^j\nabla_i\eta_j + \Delta\eta_i - \tfrac{2}{n}\nabla_i\nabla^j\eta_j = 0.\tag{5}$$

Further decomposing $\xi_i = \nabla_i\psi + \theta_i$, where ψ is a scalar and θ is a transverse vector, we arrive at decomposition (1). It follows that the decomposition $(\varphi, \psi, \theta, \omega)$ of h is unique and irreducible. For a compact Riemannian (M, g, ∇), one can further prove that the decomposition is orthogonal. The proof is a trivial extension of the proof given in Ref. [2].

Definition 2. *Let f be a scalar. Let h and h' be symmetric tensors of type $(0, 2)$ with the decompositions $(\varphi, \psi, \theta, \omega)$ and $(\varphi', \psi', \theta', \omega')$, respectively. We say that the decompositions of h and f decouple if f does not depend on θ and ω. We say that the decompositions h and h' decouple if φ' and ψ' do not depend on θ and ω, θ' does not depend on φ, ψ, and ω, and ω' does not depend on φ, ψ, and θ.*

For an infinitesimal deformation of M with the metric perturbation δg, the perturbations of the scalar curvature, the Ricci tensor, and the Einstein tensor are

$$\delta S = (\nabla^i\nabla^j - g^{ij}\Delta - R^{ij})\delta g_{ij}, \tag{6}$$

$$\delta R_{ij} = \tfrac{1}{2}(\nabla^k\nabla_i\delta g_{kj} + \nabla^k\nabla_j\delta g_{ki} - \Delta\delta g_{ij} - g^{kl}\nabla_i\nabla_j\delta g_{kl}), \tag{7}$$

$$\delta G_{ij} = \tfrac{1}{2}\big(\nabla^k\nabla_i\delta g_{kj} + \nabla^k\nabla_j\delta g_{ki} - (\Delta + S)\delta g_{ij}\big)$$
$$+ \tfrac{1}{2}\big(-g^{kl}\nabla_i\nabla_j + g_{ij}(-\nabla^k\nabla^l + g^{kl}\Delta + R^{kl})\big)\delta g_{kl}. \tag{8}$$

Theorem. *Let δg, δS, δR, and δG be infinitesimal perturbations of the metric tensor, the scalar curvature, the Ricci tensor, and the Einstein tensor, respectively, on a manifold M. The following statements are equivalent:*

1. M is an Einstein manifold.

2. The decompositions of δg and δS decouple.

3. The decompositions of δg and δR decouple.

4. The decompositions of δg and δG decouple.

Proof. Let $(\varphi, \psi, \theta, \omega)$ be the decomposition of δg. We find

$$\delta S = \tfrac{1}{n}\big((1-n)\Delta - S\big)\varphi + \tfrac{2}{n}\big((n-1)\Delta\Delta + S\Delta + n(\nabla^i R_{ij})\nabla^j\big)\psi$$
$$+ 2(\nabla^i R_{ij})\theta^j - R_{ij}\omega^{ij}, \tag{9}$$

$$\delta R_{ij} = \tfrac{1}{2n}\big((2-n)\nabla_i\nabla_j - g_{ij}\Delta\big)\varphi$$
$$+ \big((1 - \tfrac{2}{n})\nabla_i\nabla_j\Delta + \tfrac{1}{n}g_{ij}\Delta\Delta + R_{ik}\nabla_j\nabla^k + R_{jk}\nabla_i\nabla^k + (\nabla_k R_{ij})\nabla^k\big)\psi$$
$$+ R_{ik}\nabla_j\theta^k + R_{jk}\nabla_i\theta^k + (\nabla_k R_{ij})\theta^k$$
$$- \tfrac{1}{2}\Delta\omega_{ij} + \tfrac{1}{2}R_{ik}\omega_j{}^k + \tfrac{1}{2}R_{jk}\omega_i{}^k - P_{ikjl}\omega^{kl}, \tag{10}$$

$$\delta G_{ij} = \tfrac{1}{2n}(2-n)(\nabla_i\nabla_j - g_{ij}\Delta)\varphi$$
$$+ \big(\tfrac{1}{n}(n-2)(\nabla_i\nabla_j - g_{ij}\Delta)\Delta - S\nabla_i\nabla_j - \tfrac{1}{2}g_{ij}(\nabla_k S)\nabla^k$$
$$+ R_{ik}\nabla_j\nabla^k + R_{jk}\nabla_i\nabla^k + (\nabla_k R_{ij})\nabla^k\big)\psi$$
$$+ R_{ik}\nabla_j\theta^k + R_{jk}\nabla_i\theta^k + (\nabla_k R_{ij})\theta^k - \tfrac{1}{2}S(\nabla_i\theta_j + \nabla_j\theta_i) - \tfrac{1}{2}g_{ij}(\nabla_k S)\theta^k$$
$$- \tfrac{1}{2}(\Delta + S)\omega_{ij} + \tfrac{1}{2}R_{ik}\omega_j{}^k + \tfrac{1}{2}R_{jk}\omega_i{}^k + \tfrac{1}{2}g_{ij}R_{kl}\omega^{kl} - P_{ikjl}\omega^{kl}, \tag{11}$$

where P is the Riemann tensor of type $(0,4)$. Since θ is an arbitrary transverse vector and ω is an arbitrary symmetric, traceless, and transverse tensor, it follows that δS does not depend on θ and ω if and only if $R = \tfrac{1}{n}Sg$, where S is a constant. In this case we find

$$\delta S = \tfrac{1}{n}\big((1-n)\Delta - S\big)\varphi + \tfrac{2}{n}\big((n-1)\Delta + S\big)\Delta\psi. \tag{12}$$

This proves that statements 1 and 2 are equivalent.

Let $(\varphi', \psi', \theta', \omega')$ be the decomposition of δR. From $\varphi' = g^{ij}\delta R_{ij}$ we find

$$\varphi' = \tfrac{1}{n}(1-n)\Delta\varphi + \tfrac{2}{n}((n-1)\Delta\Delta + n\nabla^i R_{ij}\nabla^j)\psi + 2\nabla^i R_{ij}\theta^j. \tag{13}$$

It follows that φ' does not depend on θ if and only if $R = \tfrac{1}{n}Sg$, where S is a constant. In this case we find

$$\varphi' = \tfrac{1}{n}(1-n)\Delta\varphi + \tfrac{2}{n}((n-1)\Delta + S)\Delta\psi, \tag{14}$$

$$\psi' = \tfrac{1}{4n}(2-n)\varphi + \tfrac{1}{2n}\big((n-2)\Delta + 2S\big)\psi, \tag{15}$$

$$\theta'_i = \tfrac{1}{n}S\theta_i, \tag{16}$$

$$\omega'_{ij} = \big(-\tfrac{1}{2}\Delta + \tfrac{1}{n-1}S\big)\omega_{ij} - W_{ikjl}\omega^{kl}. \tag{17}$$

This proves that statements 1 and 3 are equivalent.

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Let $(\varphi'', \psi'', \theta'', \omega'')$ be the decomposition of δG. From $\varphi'' = g^{ij} \delta G_{ij}$ we find

$$\varphi'' = \tfrac{1}{2n}(n-1)(n-2)\Delta\varphi + \left(\tfrac{1}{n}(n-1)(2-n)\Delta\Delta - S\Delta + (1-\tfrac{n}{2})(\nabla_i S)\nabla^i \right.$$
$$\left. + 2R_{ij}\nabla^i\nabla^j\right)\psi + 2R_{ij}\nabla^i\theta^j + (1-\tfrac{n}{2})(\nabla_i S)\theta^i + \tfrac{n}{2}R_{ij}\omega^{ij}. \tag{18}$$

It follows that φ'' does not depend on θ and ω if and only if $R = \tfrac{1}{n}Sg$, where S is a constant. In this case we find

$$\varphi'' = \tfrac{1}{2n}(n-1)(n-2)\Delta\varphi + \tfrac{1}{n}(2-n)\big((n-1)\Delta + S\big)\Delta\psi, \tag{19}$$

$$\psi'' = \tfrac{1}{4n}(2-n)\varphi + \tfrac{1}{2n}(n-2)(\Delta - S)\psi, \tag{20}$$

$$\theta_i'' = \tfrac{1}{2n}(2-n)S\theta_i, \tag{21}$$

$$\omega_{ij}'' = \left(-\tfrac{1}{2}\Delta + \tfrac{3-n}{2(n-1)}S\right)\omega_{ij} - W_{ikjl}\omega^{kl}. \tag{22}$$

This proves that statements 1 and 4 are equivalent and thus concludes the proof of the theorem. $\qquad\square$

III. APPLICATION

We now apply our theorem to the theory of small perturbations in general relativity [1]. We consider an $(n + 1)$-dimensional pseudo-Riemannian space-time (M', g') of signature $(1, n)$ as locally foliated by n-dimensional Riemannian spaces $(M(t), g(t))$ which depend on the temporal coordinate t as a parameter. We use Gaussian normal coordinates on M', so that $g_{00}' = -1$, $g_{0i}' = 0$, $g_{ij}' = g_{ij}(t)$, where the index 0 refers to the coordinate t. We can always choose these coordinates in such a way that they remain locally Gaussian normal after an infinitesimal deformation of M', so that $\delta g_{00}' = 0$, $\delta g_{0i}' = 0$, $\delta g_{ij}' = \delta g_{ij}(t)$.

By the theorem, the decompositions of $\delta g(t)$ and $\delta G(t)$ decouple if and only if $M(t)$ is an Einstein manifold. Consider now the case $n = 3$, which is particularly important in cosmology. Since $W(t) = 0$ in such a case, it follows that $M(t)$ is an Einstein manifold if and only if it is a manifold of constant curvature. If $M(t)$ is a manifold of constant curvature for each t, then M' is a homogeneous, isotropic space-time. In such a case, the metric g' is essentially uniquely specified by the homogeneity and isotropy conditions, and depends on the scalar curvature $S(t)$. We thus arrive at the standard Robertson-Walker cosmological model, for which decoupling of the decompositions is well-known. The theorem implies, in particular, that such a space-time is the only case in which the decompositions of δg and

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δT decouple. For any other space-time, a perturbation δT of one type leads to multiple types of perturbations in δg. This feature should be useful in suggesting more general cosmological models [3], which agree with the recently reported observational deviations from the homogeneous, isotropic model in cosmology.

Acknowledgments

[1] See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, Oxford, 1975; S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons, New York, 1972.

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[3] For a discussion and further references, see A. Berera, R. V. Buniy and T. W. Kephart, JCAP **0410**, 016 (2004) [arXiv:hep-ph/0311233]; R. V. Buniy, A. Berera and T. W. Kephart, Phys. Rev. D **73**, 063529 (2006) [arXiv:hep-th/0511115].